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Stability diagrams for a heterogeneous ensemble of particles at the collinear libration points of the photogravitational three-body problem☆

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ABSTRACT

The stability of the collinear libration points in the photogravitational elliptical three-body problem is investigated. The distribution of the inner collinear libration points located between the principal bodies in the system is revealed. A method of finding collinear libration points for particles with specified reduction coefficients is given. Stability diagrams are constructed for an entire heterogeneous ensemble of particles (cloud) at libration points, which, in particular, make it possible to trace cloud subdivision scenarios. The characteristics (the number of clusters, the diameter of each cluster and the distances to the components of a binary system) are determined for a binary star system similar to α–Centauri.

1. Statement of the problem

In photogravitational celestial mechanics, the light pressure of a body (star) is taken into account along with the Newtonian forces of attraction.[1] In several cases the light flux is of such intensity that the light pressure force competes with the gravitational force and even exceeds it in magnitude.

The light pressure force depends not only on the luminosity of the star, but also on the characteristics (geometric dimensions, density and reflectivity) of the specific particle. The relation between the parameters of the star and the particle gives the reduction coefficient

$$Q = 1 - (1 + \varepsilon)A\frac{E}{fM} \tag{1.1}$$

(fM is the gravitational parameter of the star, E is a coefficient that characterizes the power of a radiation source, A is the area-mass ratio of the particle, and ε is the reflectivity). For each particle, the reduction coefficient Q has a constant value and characterizes the degree of susceptibility of the particle to the radiation. The gravitational force of a star has the strongest effect on fairly large and very dense particles with small values of A and ε, and, therefore, $Q > 0$. For very small particles with a high area-mass ratio and a high reflectivity, the effect of light exceeds the effect of gravity ($Q < 0$).

The photogravitational three-body problem, which was introduced by V. V. Radziyevskii[2] and provides a way of taking into account the light repulsion factor in the fundamental problem[3] of celestial mechanics, has become a good dynamical model for studying the motion of microparticles in a system similar to a Sun–planet–particle system, as well as in a repulsive gravitational field of binary star systems.

As in the classical three-body problem,[4] to describe the motion of a particle within the photogravitational three-body problem it is convenient to use equations in a rotating system of coordinates when Nechvile's transformation is taken into account. Then the corresponding function

$$\begin{aligned} W &= \frac{1}{2}(x^2 + y^2) - \frac{1}{2}e\cos\upsilon z^2 + Q_1\frac{1-\mu}{R_1} + Q_2\frac{\mu}{R_2} \\ R_1 &= \sqrt{(x+\mu)^2 + y^2 + z^2}, \quad R_2 = \sqrt{(x-1+\mu)^2 + y^2 + z^2} \end{aligned} \tag{1.2}$$

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($1-\mu$ and μ are the dimensionless masses of the principal bodies, υ is the true anomaly, e is the eccentricity of the orbit of the principal bodies, and R_1 and R_2 are the distances from the particle with the coordinates x, y, z to the principal bodies) contains two reduction coefficients: Q_1 and Q_2.[1] According to their physical meaning, the numerical values of Q_1 and Q_2 do not exceed unity, while in the classical three-body problem $Q_1 = 1$, and $Q_2 = 1$ (there is no emission from the bodies).

The libration points, that is, the constant solutions of the system of dynamical equations adopted, are relative equilibria in the circular problem and periodic motions in the elliptical problem. They are found from the system of equations

$$\frac{\partial W}{\partial x} \equiv x - Q_1\frac{(1-\mu)(x+\mu)}{R_1^3} - Q_2\frac{\mu(x+\mu-1)}{R_2^3} = 0$$
$$\frac{\partial W}{\partial y} \equiv y\left(1 - Q_1\frac{1-\mu}{R_1^3} - Q_2\frac{\mu}{R_2^3}\right) = 0, \quad \frac{\partial W}{\partial z} = 0 \tag{1.3}$$

The collinear libration points are located on the straight line connecting the principal bodies. For them $y = 0$ and $z = 0$. Their positions on the x axis are determined from the first equation in system (1.3). The corresponding equation for calculating the x coordinate of a collinear libration point has the form

$$f(x) \equiv x - Q_1\frac{(1-\mu)(x+\mu)}{|x+\mu|^3} - Q_2\frac{\mu(x+\mu-1)}{|x+\mu-1|^3} = 0 \tag{1.4}$$

Equation (1.4) contains the parameters μ, Q_1 and Q_2, which are mutually independent in the mathematical statement of the problem. Therefore, the photogravitational problem under consideration admits of a three-parameter family of collinear libration points. Consequently, we obtain a stability problem in a three-dimensional parametric space. When we change to the elliptical problem, the number of parameters increases to four.

Below we will solve the problem of finding generalized parameters for analysing the stability of collinear libration points that are such that the dimension of the parameter space would be minimal and the information about the stability would still be adequate. The results of the investigation are stability diagrams constructed in the (a, e) plane (a is a certain generalized parameter), which give such information regarding the stability of the collinear libration points at once for all values of the parameters of the problem. The distribution of the inner libration points located between the principal bodies on the x axis is also obtained, and a simple method for finding all the collinear libration points of a particle with specified reduction coefficients is given.

It should be noted that the stability problem will be solved below in the linear approximation.

2. Location of the collinear libration points

We next consider the case in which both components of the binary star emit radiation.

From the expression for the reduction coefficient (1.1) it follows that the relation which holds for the components of a binary star has the form[5]

$$K = \frac{1-Q_2}{1-Q_1} = C\frac{1-\mu}{\mu} \tag{2.1}$$

in which the parameter $C = E_2/E_1$ characterizes the ratio between the radiated powers of the components, and the coefficient K takes positive values. The expression on the right-hand side of equality (2.1) depends only on the parameters of the binary star. For a specific binary star system the coefficient K is equal to the known number K^*; therefore, equality (2.1) relates the reduction coefficients Q_1 and Q_2 linerly.

From the foregoing we find that for a given binary star system Eq. (1.4), which specifies the libration points, contains only one parameter, which can chosen to be either Q_1 or Q_2. Therefore, the collinear libration points form one-parameter families.

Instead of the reduction coefficient Q, it is sometimes convenient to use the number $q = 1 - Q$, which is equal to the ratio of the light pressure force to the gravitational attraction force. We write the form of Eq. (1.4) that will specify the collinear libration points using the coefficients q_1 and q_2, which are negative in physical problems (and positive in the case of emission from both components). Then, taking an interest only in the inner collinear libration points and taking into account the equality $q_2 = Kq_1$, we obtain

$$f_0(x) \equiv x - \frac{1-\mu}{(x+\mu)^2} + \frac{\mu}{(x+\mu-1)^2} = -q_1 g(x)$$
$$g(x) \equiv \frac{1-\mu}{(x+\mu)^2} - K\frac{\mu}{(x+\mu-1)^2} \tag{2.2}$$

The function $f_0(x)$ is well-known from the classical three-body problem:[3] the equation $f_0(x) = 0$ specifies a single inner collinear libration point, whose coordinate is denoted here by x^*. In the interval $-\mu < x < x^*$ the function $f_0(x)$ is negative, and in the interval $x^* < x < 1 - \mu$ it is positive.

The single null of the function $g(x)$ equals

$$x^{**} = -\mu + \frac{\sqrt{1-\mu}}{\sqrt{K\mu} + \sqrt{1-\mu}}$$

Here $g(x) > 0$ if $x < x^{**}$, and $g(x) < 0$ if $x > x^{**}$.

In the case of a binary star, we have $q_1 > 0$. Therefore, Eq. (2.2) has roots in ranges that do not contain the interval (x^*, x^{**}) if $x^{**} > x^*$ and do not contain the interval (x^{**}, x^*) in the case when $x^* > x^{**}$. Note that in the case of $x^* = x^{**}$, we obtain $q_1 = 0$ (the body does not emit radiation).

Now we take any number x that lies in the interval $(-\mu, 1-\mu)$ but does not belong to the excluded interval, and we calculate the value of q_1, which is found to be positive. Proceeding in this manner, we obtain all the libration points and the corresponding values of q_1. Then, from the graph of the function

$$q_1 = \frac{x(x+\mu)^2(x+\mu-1)^2 - (1-\mu)(x+\mu-1)^2 + \mu(x+\mu)^2}{-(1-\mu)(x+\mu-1)^2 + K(x+\mu)^2} \tag{2.3}$$

it follows that $q_1 \in (0, +\infty)$.

The discussion takes into account the presence of the parameters μ and C, which characterize a binary star system. Therefore, we obtain the following assertion.

Theorem. The inner collinear libration points of the photogravitational three-body problem in the case of two emitting bodies fill the entire interval $(-\mu, 1-\mu)$ between the principal bodies except the interval between the points x^* and x^{**}. The values of the reduction coefficients of the particle that correspond to the collinear libration points are found from relation (2.3).

Remarks. 1°. In the case of emission from only one body, $K = 0$ must be taken in relation (2.2), the theorem remains valid, and only the collinear libration points fill the entire interval between the principal bodies.

2°. The case of collinear libration points located outside the interval $(-\mu, 1-\mu)$ can be analysed in a similar manner.

From the above discussion we obtain a very simple method for finding all the inner collinear libration points. Here there is no need to solve Eq. (1.4), all the collinear libration points are already known, and all that remains is to use formula (2.3) to find the corresponding reduction coefficients of the particle: $Q_1 = 1 - q_1$ and $Q_2 = 1 - Kq_1$.

Another aspect of the problem relates to the number of collinear libration points that have the same reduction coefficients. Such a statement of the problem for determining libration points inherits the traditions of the investigation of the classical three-body problem.

Suppose Q_1 and Q_2 are specified. Then it is known[6] that there are three collinear libration points (L_1, L_2 and L_3) in the case of positive reduction coefficients. When Q_1 and Q_2 are negative, the outer libration points L_2 and L_3 vanish, and two more inner collinear libration points appear. When Q_1 and Q_2 are negative, the collinear libration points are denoted by L_{11}, L_{12} and L_{13}, If the reduction coefficient of one of the components of a binary star is positive and the other has a negative reduction coefficient, the collinear libration point that is outer relative to the component with a reduction coefficient greater than zero is absent.[6]

For a specified binary star system (when the parameters μ and K are specified), from the graph of the function $q_1(x)$ we quickly obtain not only the distribution of all the inner collinear libration points along x, but also their number and coordinates for fixed q_1. For this it is sufficient to draw the straight line $q_1 = \text{const}$ in the (x, q_1) plane and to find the points of intersection of the graph of $q_1(x)$ with the straight line. The intersection points give the collinear libration points required.

3. Stability diagrams

It seemed[7,8] that collinear libration points inherit the property of instability that occurs in the classical three-body problem. This conclusion was contested, and the possibility of the stability of an inner collinear libration point when Q_1 and Q_2 are positive was demonstrated.[9] In addition, a collinear libration point can also be stable when the reduction coefficients are negative,[10] and stability regions were constructed[10] in the circular problem in the (Q_1Q_2) plane with fixed values of the mass parameter μ. It was found that the inner collinear libration point will be stable.

The investigations in Refs 7–10 were performed for the circular problem using the conventional approach for the three-body problem: the positions of the libration points were determined for fixed values of the parameters μ, Q_1 and Q_2, and then stable (unstable) solutions were found. The problems arising with the dimension of the parametric space were overcome by indicating the algorithm for constructing the stability region for a fixed value of the mass parameter μ. Subsequently,[11] an approach with a transition to phase space, which proved itself in the investigation of triangular libration points,[12] was applied to the problem of the stability of collinear libration points. Taking into account the fact that for each fixed pair of principal bodies the reduction coefficients of the particle (Q_1 and Q_2) are related by a definite linear relationship[5] makes renders the parametric space two-dimensional.

The stability has previously been investigated in the elliptical problem.[13–17] The non-linear problem was considered,[14] but results were given only for special values of the parameters ($Q_1 = Q_2 = \underline{Q}$, $Q = 1, 0.98, 0.95$). The weakly elliptical problem was investigated,[15] and results from the numerical construction of the stability regions were presented.[13,16,17]

The starting point for obtaining stability diagrams was Ref. 15, where, firstly, it was shown (for $Q_{1,2} > 0$) that parametric resonance is the only factor that leads to instability of the collinear libration points in the weakly elliptical problem and, secondly, the investigation was performed using the generalized parameter

$$a = Q_1 \frac{1-\mu}{|x+\mu|^3} + Q_2 \frac{\mu}{|x+\mu-1|^3} \tag{3.1}$$

The parameter a naturally appears initially in the problem of determining the number of inner collinear libration points for fixed $Q_{1,2}$. In fact, from expression (1.4) we obtain $f'(x) = 1 + 2a(x)$. Therefore, for example, in the case of $Q_{1,2} > 0$, we hence quickly derive the existence of only one inner collinear libration point. It turns out[15] that the variational equations for the collinear libration points contain only the one parameter a:

$$X'' - 2Y' - \frac{1+2a}{1+e\cos\nu}X = 0, \quad Y'' + 2X' - \frac{1-a}{1+e\cos\nu}Y = 0 \tag{3.2}$$

A formal mathematical investigation of the stability of the zero solution of system (3.2) in the (a,e) parameter plane was previously peformed.[17] However, the legitimate question of whether the real stable collinear libration point in the photogravitational three-body problem corresponds to a stable point in the (a,e) plane and whether, conversely, each stable collinear libration point has a corresponding stable point in the (a,e) plane arises. We can speak about a stability diagram in the (a,e) plane only after there is an answer to this question.

System (3.2) is reversible, as is shown by the invariance of the system to the substitution $(X,Y) \to (X,-Y)$. Therefore, the corresponding theory for reversible mechanical systems, particularly the theory of parametric resonance,[18] will be used in the subsequent treatment.

In the circular problem the characteristic equation of system (3.2) has the form

$$\lambda^4 + (2-a)\lambda^2 - (1-a)(1+2a) = 0 \tag{3.3}$$

and its roots are:

$$\lambda_\alpha^2 = \frac{1}{2}\left(a - 2 \pm \sqrt{(9a-8)a}\right), \quad \alpha = 1,2$$

An elementary analysis shows that in the intervals $8/9 < a \le 1$ and $-1/2 < a \le 0$ the roots λ_α are pure imaginary (on the boundaries they are multiple); therefore, these values of the parameter a belong to the region of necessary conditions for stability.

When the values of the parameters are small, instability zones may appear in the stability region as a consequence of parametric resonance. The pure imaginary roots of Eq. (3.3) then satisfy either of the conditions

$$\lambda_\alpha = pi, \quad \alpha = 1,2 \quad \text{or} \quad \lambda_1 + \lambda_2 = pi, \quad p = 1,2,\ldots$$

We will use a^* to denote the values of the parameter a at which the resonance condition is satisfied. Then, in the interval $8/9 < a \le 1$ there is one single-frequency parametric resonance

$$2\lambda_1 = i \; a^* = \left(\frac{1}{16}(5+\sqrt{97}) = 0.928\ldots\right)$$

and in the interval $-1/2 < a \le 0$ there are two such parametric resonances

$$2\lambda_1 = i \left(a^* = \frac{1}{16}(5-\sqrt{97}) = -0.303\ldots\right)$$

$$2\lambda_1 = 3i \left(a^* = \frac{1}{16}(13-\sqrt{369}) = -0.388\ldots\right)$$

In the latter interval there is also a two-frequency resonance

$$-\lambda_1 + \lambda_2 = i; \; |\lambda_1| = 0.457\ldots, \quad |\lambda_2| = 1.457\ldots, \; a^* = -1/3$$

The parametric resonances revealed are the only fact which leads to the conclusion that in the weakly elliptical problem the inner collinear libration points are stable if $a \neq a^*$. Explicit formulae for the coefficient that gives instability for single-frequency parametric resonance were previously obtained.[15]

The stability regions of system (3.2) were constructed numerically for values of the eccentricity $e > 0$. They are shown shaded in Fig. 1 (the diagram on the left corresponds to the case in which gravitation predominates, and the diagram on the right corresponds to the case

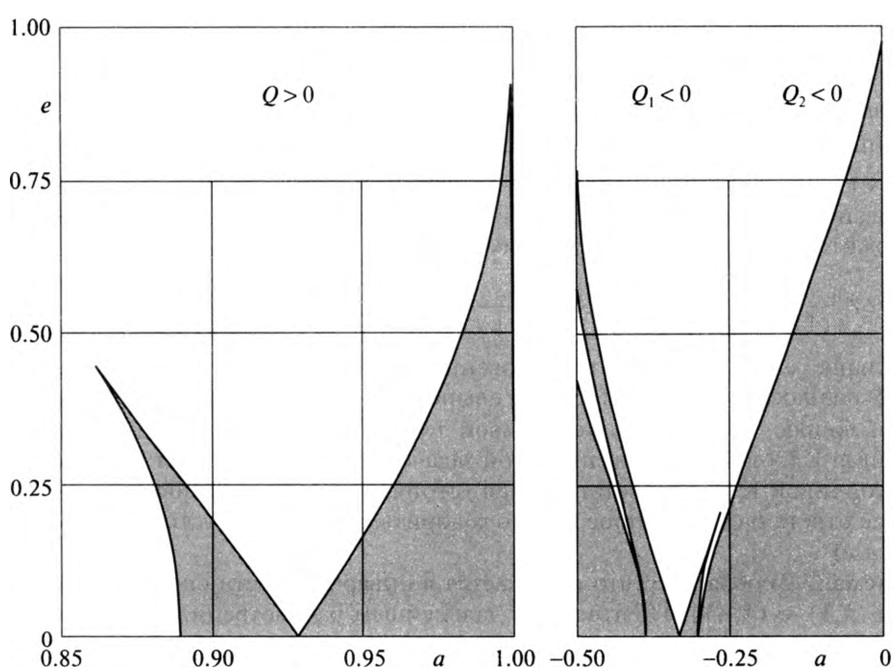

Fig. 1.

in which repulsion predominates). Instability "wedges," which border on the horizontal axis and are caused by parametric resonances, are seen. When $Q_{1,2} > 0$, the single wedge corresponds to the resonance at $2\lambda_1 = i$.[15] In the case when $Q_1 < 0$, $Q_2 < 0$, the single-frequency resonances at $2\lambda_1 = i$ and $2\lambda_1 = 3i$ correspond to the narrow instability wedges on the diagram, while the two-frequency resonance at $-\lambda_1 + \lambda_2 = i$ produces the broad instability region that borders on the horizontal axis at $a = -1/3$.

According to the theorem proved in Section 2, the inner collinear libration points fill the entire gap $(-\mu, 1-\mu)$ between the principal bodies except the interval between the points x^* and x^{**}, and the values of the coefficient q_1 ($q_2 = Kq_1$) corresponding to them are found using formula (2.3). Therefore, after specifying the coordinate of a collinear libration point, we calculate the generalized parameter a using formula (3.1). Then we single out the collinear libration points that are stable to the parameter a by numerically solving the Cauchy problem for system (3.2). As a result, we obtain the stable collinear libration points in the phase space.

The set of values of q_1 for the collinear libration points fill the positive semiaxis. This fact does not depend on the specific values of the parameters μ and K of the binary star system. Since $q_1 \in (0, +\infty)$, Q_1 and Q_2 take all possible values in the interval $(-\infty, 1)$. We also take into account that the values of the x coordinate of the collinear libration points in formula (3.1) belong to the intervals indicated in the theorem proved in Section 2. Therefore, from expression (3.1) we find that the values of the parameter a cover the entire numerical axis, regardless of the values of μ and K chosen. Then the stable collinear libration points identified using the parameter a are also maintained when μ and K are varied, since system (3.2), which does not depend on μ and K, is used to find them. The required substitution of specific numerical values of μ and K is used to determine the coordinates of the stable collinear libration points in the phase space.

The discussions have been conducted for a fixed eccentricity e of the orbit; therefore, we will vary e and construct the stability regions already in the (a, e) plane.

Thus, the stability regions shown in Fig. 1 provide complete information on the stable collinear libration points of a binary star system, regardless of the specific parameters of the system.

We will use the stability diagrams shown in Fig. 1 to resolve the question of the existence of clusters of microparticles (clouds) and to evaluate their diameter and their distances from the components of a binary star for which the eccentricity of the orbit, the masses of the components, and the radiated power of their light are known. However, an analysis of the diagrams immediately enables us to arrive at the unexpected qualitative conclusion that there are several clouds and even enables us to determine their number precisely.

In fact, it can be seen that when the components of the binary star move in a circular orbit, there are only two clouds: one at $Q_1 > 0$, $Q_2 > 0$ (the left-hand part of the figure), and the other at $Q_1 < 0$, $Q_2 < 0$ (the right-hand part). When the star moves to a weakly elliptical orbit, regions that do not contain "stable" particles appear on each diagram, and the cloud is thereby divided into two clouds (the left-hand part of the figure) or four clouds (the right-hand part). This is revealed, if the straight line $e = \text{const} << 1$ is drawn on the diagram. Similar straight lines corresponding to $e = \text{const}$ (for specific values of e) that have segments in the shaded regions give the number of clouds for a specific value of e.

It follows from the diagram in the left-hand part of the figure that the scenario of cloud formation in the case when $Q_1 > 0$, $Q_2 > 0$ is fairly simple. At first, for $e = 0$ we have one cloud, which divides into two clouds on passing into the region $e > 0$. These clouds exist up to a certain value $e = 0.44255\ldots$, and when e is increased further, the left-hand cloud vanishes, and only one cloud remains. The parallel scenario is more complicated in the case when $Q_1 < 0$, $Q_2 < 0$ (the right-hand part of the figure). Here the number of clouds increases from one to four and then decreases to one as e increases.

4. Calculations for the binary star system α-Centauri

We will use the stability diagram to determine the characteristics of clusters of microparticles in a binary star system whose characteristics are similar to the characteristics of the binary star α-Centauri. This system contains two components that are similar to the Sun. Their masses are $M_1 = 0.907\, M_S$ and $M_2 = 1.095\, M_S$, and they have radiation powers $E_1 = 1.56 L_S$ and $E_2 = 0.5 L_S$, respectively (M_S and L_S are the mass and radiation power of the Sun). The components revolve about the centre of mass in elliptical orbits with eccentricity $e = 0.519\ldots$.

We calculate the values of the parameters assumed here:

$$\mu = M_2/(M_1 + M_2) = 0.45304695\ldots \text{ and } C = E_2/E_1 = 3.12\ldots.$$

Next, we use the stability diagrams (Fig. 1), and for $e = 0519\ldots$ we find the gaps a that correspond to stable libration points. These points form two sets, i.e., clusters. For the first cluster, which is formed by the stable collinear libration points L_1 we have $a \in [0.9847\ldots, 1]$, $Q_1 > 0$, $Q_2 > 0$, and for the second cluster, which is formed by the stable points L_{12}, we obtain $a \in [-0.14047\ldots, 0]$.

We determine the x coordinates of the points L_1 and L_{12} on the phase plane by solving Eq. (1.4). For this purpose, we use expression (3.1) and take into account formula (2.1) to find the value of Q_1 as a function of x and the parameters μ, C and a, and we substitute this value into Eq. (1.4). As a result, we obtain a seventh-order polynomial in x, whose roots we find for each fixed value of a and the values of μ and C corresponding to the α-Centauri system.

As a result, we obtain the characteristics of the hypothetical clusters (clouds) in the α–Centauri system presented in Table 1. The left-hand ($x^{(1)}$) and right-hand ($x^{(2)}$) boundaries of each cluster along the x coordinate, the diameter D of the cluster, and the distance R_2 from the cluster to the second principal body are given here. All the linear dimensions are given in kilometres after Nechvile's transformation is taken into account. The distance between the two clouds is equal to 125.96×10^6 km.

Particles with different characteristics can be found at a stable collinear libration point of a binary star. In fact, for a specified value of the eccentricity e from the stability region, we can find the values of the parameter a that correspond to stability, and we can calculate the

Table 1

CLP	$x^{(1)}$	$x^{(2)}$	$D \cdot 10^{-6}$	$R_2 \cdot 10^{-6}$
L_1	0.44932...	0.45013...	2.87414...	205.95...
L_{12}	0.48566...	0.48886...	11.37762...	343.29...

x coordinate of the point L_1 or L_{12}. According to formula (2.2), this value of x corresponds to the fully specified reduction coefficient q_1. Then substituting $Q = Q_1 = 1 - q_1$ into formula (1.1), we find that the particles located at a stable collinear libration point obey the inverse proportionality law $A = k/(1 + \varepsilon)$, where $k = \text{const}$. The formula $A = 3/(4\pi\delta r)$ is valid for a homogenous particle of radius r and density δ.

When the data presented in Table 1 are taken into account, the possibility of finding a particle with different characteristics at a specific stable collinear libration point and the fact that the parameter a belongs to the stability region for the entire set of stable collinear libration points indicate that gigantic stable heterogeneous ensembles of material (clouds) can form in a binary star system.

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References

1. Razdiyevskii VV. *Photogravitational Celestial Mechanics*. Nizhnii Novgorod: Izd Yu. A. Nikolayev; 2003.
2. Razdiyevskii VV. The Restricted three-body problem taking radiation pressure into account. *Astron Zh* 1950;**30**(4):249–56.
3. Szebehely V. *Theory of Orbits. The Restricted Problem of Three Bodies*. New York, London: Academic Press; 1967.
4. Markeyev AP. *Libration Points in Celestial Mechanics and Space Dynamics*. Moscow: Nauka; 1978.
5. Kunitsyn AL. The stability of the triangular libration points in the photogravitational three-body problem. *Prikl Mat Mekh* 2000;**64**(5):788–94.
6. Luk'yanov LG. Lagrange solutions in the restricted photogravitational circular three-body problem. *Astron Zh* 1984;**61**(3):564–70.
7. Schuerman DW. The restricted three-body problem including radiation pressure. *Astrophys J* 1980;**238**(1):337–42.
8. Schuerman DW. The effect of radiation pressure on the restricted three-body problem. *Solid Part Solar Syst* 1980;(90):285–8.
9. Kunitsyn AL, Tureshbayev AT. Collinear libration points of the photogravitational three-body problem. *Pis'ma Astron Zh* 1983;**9**(7):432–5.
10. Luk'yanov LG. The stability of the Lagrange points in the restricted photogravitational three-body problem. *Astron Zh* 1986;**63**(6):1222–9.
11. Kunitsyn AL. The stability of the collinear libration points of the photogravitational three-body problem. *Prikl Mat Mekh* 2001;**65**(4):720–4.
12. Kunitsyn AL, Tureshbayev AT. The stability of the triangular libration points of the photogravitational three-body problem. *Pis'ma Astron Zh* 1985;**11**(2):145–8.
13. Luk'yanov LG, Kochetkova AYu. The stability of the Lagrange libration points in the restricted photogravitational elliptical three-body problem. *Vestnik MGU Ser 3 Fiz Astron* 1996;(5):71–6.
14. Kochetkova AYu. The stability of the triangular libration points in the non-linear approximation in the three-dimensional restricted photogravitational elliptical three-body problem. *Vestnik MGU Ser 3 Fiz Astron* 1999;(5):69–71.
15. Tkhai VN. Parametric resonance in the problem of the stability of the collinear libration points of the photogravitational three-body problem. In: *Problems in the Investigation of the Stability and Stabilization of Motion*. Moscow: VTs RAN; 2001; Part 2: 112–21.
16. Zimovshchikov AS, Tkhai VN. Instability of the libration points and resonance phenomena in the restricted photogravitational elliptical three-body problem. *Astron Vestn ik* 2004;**38**(2):180–90.
17. Zimovshchikov AS, Tkhai VN. The stability of the collinear libration points in the restricted photogravitational elliptical three-body problem. In: The *Fourth Polyakhov Lectures*. St. Petersburg: Izd VVM; 2006: 112–9.
18. Tkhai VN. Some problems on the stability of a reversible system with a small parameter. *Prikl Mat Mekh* 1994;**58**(1):3–12.

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